Basic Concepts of Dirac Notation

Exercise 1. Write the following **quantum states** as vectors.

a) $|\psi_a\rangle = \frac{1}{\sqrt{2}}\left(|\phi_1\rangle + |\phi_2\rangle\right)$

b) $|\psi_b\rangle = i|\chi_2\rangle \quad \{|\chi_i\rangle\}, i = 1\cdots 3$

c) $\langle\psi_b| = |\psi_b\rangle^\dagger$

Exercise 2. Write the **bra** $\langle\psi|$ for the **ket** $|\psi\rangle$ with components:

$$|\psi\rangle = \begin{pmatrix} i \\ 0 \\ 0 \end{pmatrix}$$

Exercise 3. Given two states $|\psi\rangle$ and $|\phi\rangle$, express their **inner product** $\langle\phi|\psi\rangle$ in terms of their components if:

$$|\psi\rangle = \begin{pmatrix} i \\ 0 \\ 0 \end{pmatrix} \text{ and } |\phi\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Exercise 4. For the same $|\psi\rangle$ and $|\phi\rangle$ of Exercise 3, express their **outer product** $|\psi\rangle\langle\phi|$ in terms of their components.

Exercise 5. Given ket $|\psi\rangle$ with components

$$|\psi\rangle = i\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

a) Is it **normalized**?

b) If not, normalize it.

Exercise 6. Prove that $|\psi_1\rangle$ and $|\psi_2\rangle$ with components

$$|\psi_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |\psi_2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

form a **basis** that is **orthogonal**, **orthonormal**, and **complete**.

Exercise 7. For $|\psi_1\rangle$ and $|\psi_2\rangle$ defined in Exercise 6, compute

$$|\Phi\rangle = \frac{1}{\sqrt{2}}\left(|\psi_1\rangle + |\psi_2\rangle\right)$$

Exercise 8. Consider the **operator** $\hat{O}$ and the ket $|\psi\rangle$ given as

$$\hat{O} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}, \quad |\psi\rangle = \frac{i}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Give $\hat{O}|\psi\rangle$.

Exercise 9. With the definitions in Exercise 8, give the **expected value** of $\hat{O}$

$$E = \left\langle \psi \middle| \hat{O} \middle| \psi \right\rangle$$

Exercise 10. Consider that $|\psi(t)\rangle$ **evolves in time** according to

$$|\psi(t)\rangle = e^{i\hat{H}t}|\psi(0)\rangle$$

where $\hat{H}$ is the **Hamiltonian operator**. If

$$\hat{H} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}, \quad |\psi(0)\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

estimate $\left|\psi(1)\right\rangle$ to the first order.

Exercise 11. Consider

$$\left|\Phi\right\rangle = \frac{1}{\sqrt{2}}\left(\left|\psi_1\right\rangle + \left|\psi_2\right\rangle\right)$$

Express $\left|\Phi\right\rangle$ in terms of these other two orthogonal vectors $\left|\varphi_1\right\rangle$ and $\left|\varphi_2\right\rangle$. Suppose that $\left|\varphi_1\right\rangle$ and $\left|\varphi_2\right\rangle$ form a **complete basis** such that $\left|\varphi_1\right\rangle\left\langle\varphi_1\right| + \left|\varphi_2\right\rangle\left\langle\varphi_2\right| = \mathrm{I}$.

Exercise 12. Consider

$$\left|\Phi\right\rangle = \sum_i a_i \left|\psi_i\right\rangle$$

Express $\left|\Phi\right\rangle$ in terms of the **orthonormal basis** $\left\{\left|\varphi_i\right\rangle\right\}$. Suppose that $\left\{\left|\varphi_i\right\rangle\right\}$ form a complete basis such that $\sum_i \left|\varphi_i\right\rangle\left\langle\varphi_i\right| = \mathrm{I}$.

Exercise 13. Consider

$$\left|\Phi\right\rangle = \int a(x)\left|x\right\rangle dx$$

Express $\left|\Phi\right\rangle$ in terms of the orthogonal basis $\left\{\left|\xi\right\rangle\right\}$. Suppose that $\left\{\left|\xi\right\rangle\right\}$ form a complete basis such that $\int \left|\xi\right\rangle\left\langle\xi\right| d\xi = \mathrm{I}$.

Exercise 14. Consider the state

$$\left|\Phi\right\rangle = \frac{1}{\sqrt{2}}\left(\left|\psi_1\right\rangle + \left|\psi_2\right\rangle\right)$$

Give the **probability** of occurrence of output ψ_1 in an experiment, supposing that $\left|\psi_1\right\rangle$ and $\left|\psi_2\right\rangle$ forms an orthonormal basis.

Exercise 15. Consider the state

$$|\Phi\rangle = \sum_i a_i |\psi_i\rangle$$

Give the probability of occurrence of output ψ_1 in an experiment, supposing that $\{|\psi_i\rangle\}$ is an orthonormal basis.

Exercise 16. Consider the state

$$|\Phi\rangle = \int a(x)|x\rangle dx$$

Give the probability of occurrence of output $x = r$ in an experiment, supposing that $\{|x\rangle\}$ is an orthonormal basis.

Exercise 17. The state of a system is described by the basis $|\psi_1\rangle$ and $|\psi_2\rangle$ with components

$$|\varphi_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |\varphi_2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

A second system is described by the basis

$$|\varphi_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |\varphi_2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Use a **tensor product** to build a basis to describe the two systems together.

Exercise 18. Consider the state

$$|\psi\rangle = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

Build the **density matrix** $\hat{\rho} = |\psi\rangle\langle\psi|$ for this state.

Exercise 19. For the state $|\psi\rangle$ of Exercise 18, show that the coherences satisfy $\rho_{12} = \rho_{21}^*$.

Exercise 20. For the state $|\psi\rangle$ of Exercise 18, show that the populations satisfy $\rho_{11}+\rho_{22}=1$.

Exercise 21. Determine the density matrix for the state $|\psi\rangle=\sum_i c_i|\phi_i\rangle$.

Exercise 22. Consider the density matrix of Exercise 21.

a) Determine the population of state 2.

b) Determine the coherence between states 1 and 2.

Assume that $\{|\phi_i\rangle\}$ is orthonormal.

Exercise 23. Consider the operation trace:

$$Tr\left[\hat{O}\right]=\sum_i \langle\phi_i|\hat{O}|\phi_i\rangle$$

Consider also the basis $\{|o_i\rangle\}$ of eigenstates of $\hat{O}$

$$\hat{O}|o_i\rangle=o_i|o_i\rangle$$

Show that if the trace is taken on the basis $\{|o_i\rangle\}$, we have

$$Tr\left[\hat{\rho}\hat{O}\right]=\langle\hat{O}\rangle$$

Exercise 24. Consider the state:

$$|\Psi\rangle=\frac{1}{\sqrt{2}}\left(|a_1\rangle|b_1\rangle+|a_2\rangle|b_2\rangle\right).$$

Verify that

$$\hat{\rho} = \frac{1}{2}\sum_{i,j=1}^{2} |a_i\rangle\langle a_j| \otimes |b_i\rangle\langle b_j|$$

Exercise 25. Consider the density matrix of Exercise 24:

$$\hat{\rho} = \frac{1}{2}\sum_{i,j=1}^{2} |a_i\rangle\langle a_j| \otimes |b_i\rangle\langle b_j|.$$

Consider also the state $|b_i\rangle$ can be expanded in terms of an orthonormal basis $\{|\phi_k\rangle\}$ such that

$$|b_i\rangle = \sum_k c_{ik} |\phi_k\rangle.$$

Calculate the **reduced density matrix** of subsystem A by tracing over system B:

$$\hat{\rho}_A = Tr_B[\hat{\rho}] = \sum_k \langle\phi_k|\hat{\rho}|\phi_k\rangle$$

and show that

$$\hat{\rho}_A = \frac{1}{2}\left(|a_1\rangle\langle a_1| + |a_1\rangle\langle a_2| + |a_2\rangle\langle a_1|\langle b_1|b_2\rangle + |a_2\rangle\langle a_2|\langle b_2|b_2\rangle\right)$$