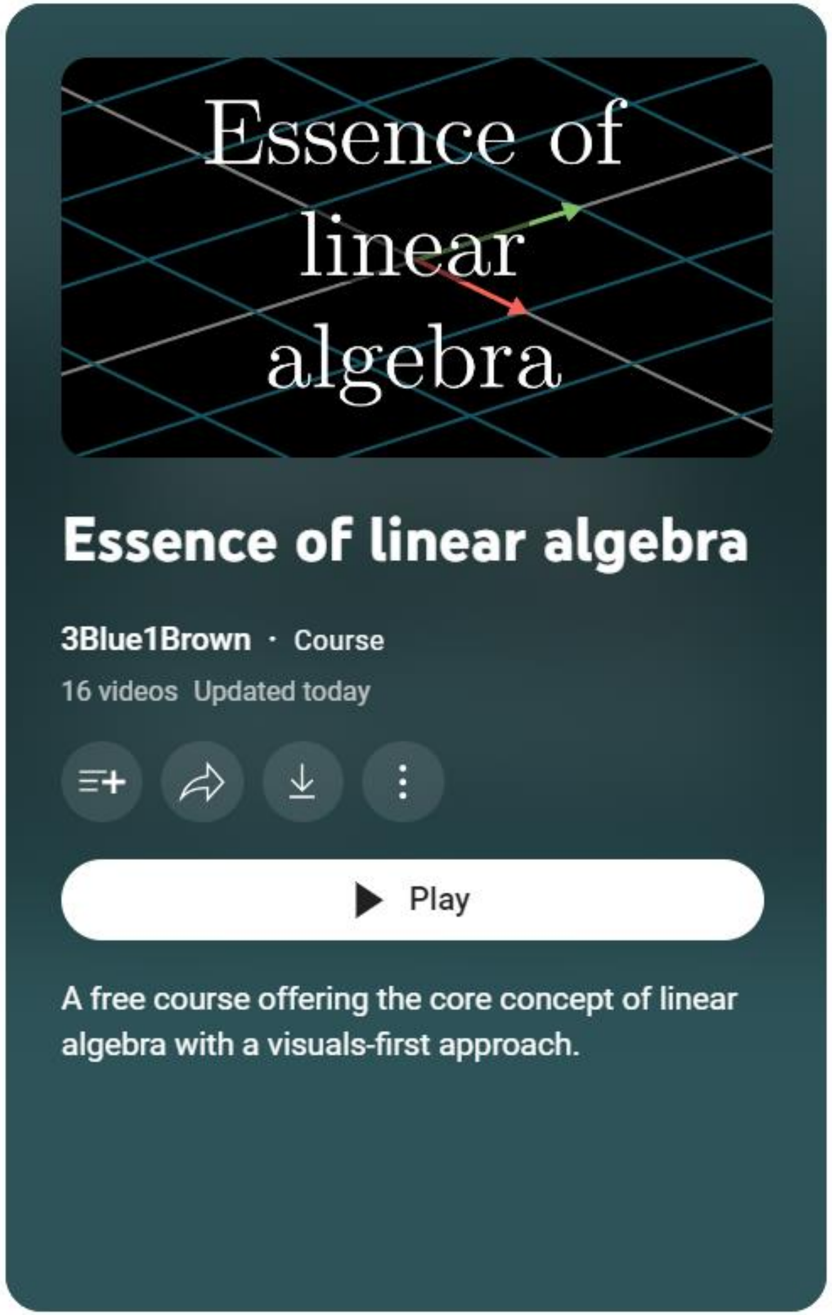
TD2 – Math review

Basic mathematics for quantum mechanics

Basic vector operation

Strongly recommended: Linear algebra course by 3Blue1Brown



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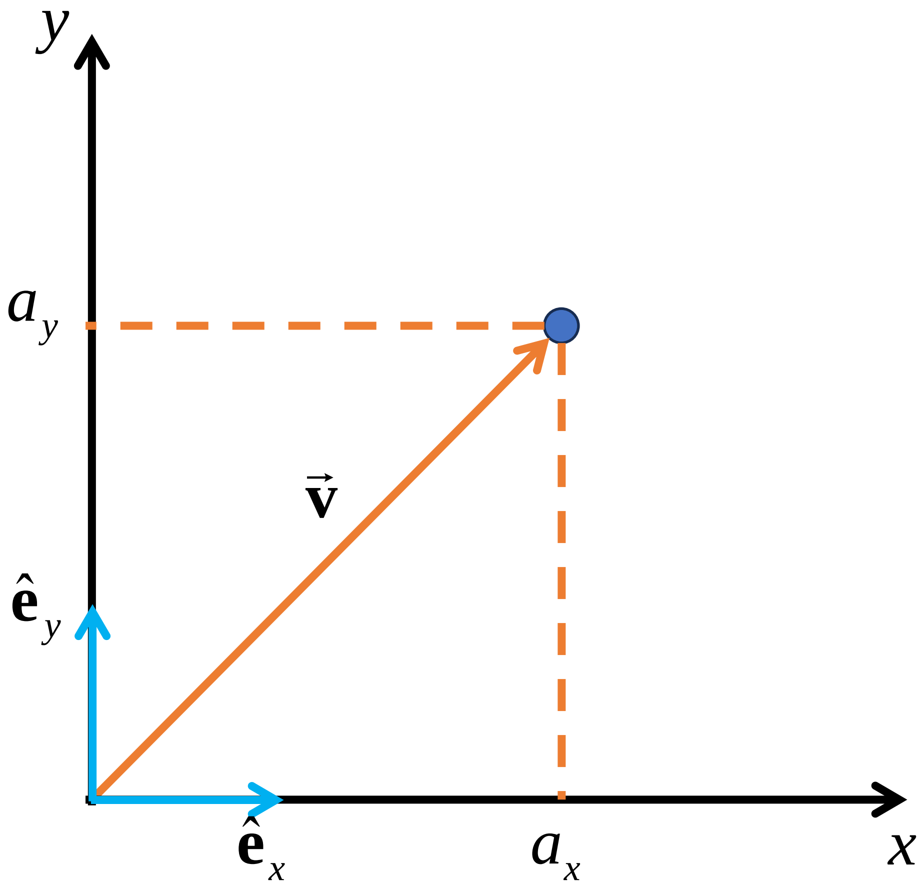
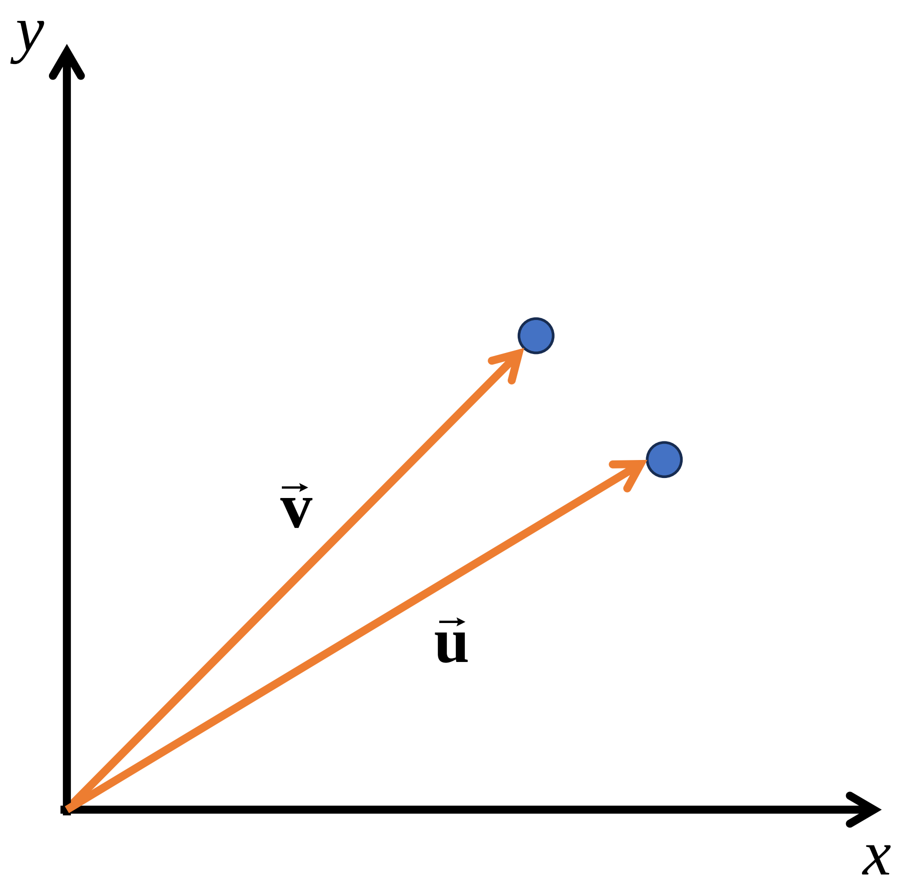


$$\vec{\mathbf{v}} = a_x \hat{\mathbf{e}}_x + a_y \hat{\mathbf{e}}_y$$

$$= \begin{pmatrix} a_x \\ a_y \end{pmatrix}$$

$$\hat{\mathbf{e}}_x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\hat{\mathbf{e}}_y = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$



$$\vec{\mathbf{v}} = a_x \hat{\mathbf{e}}_x + a_y \hat{\mathbf{e}}_y$$

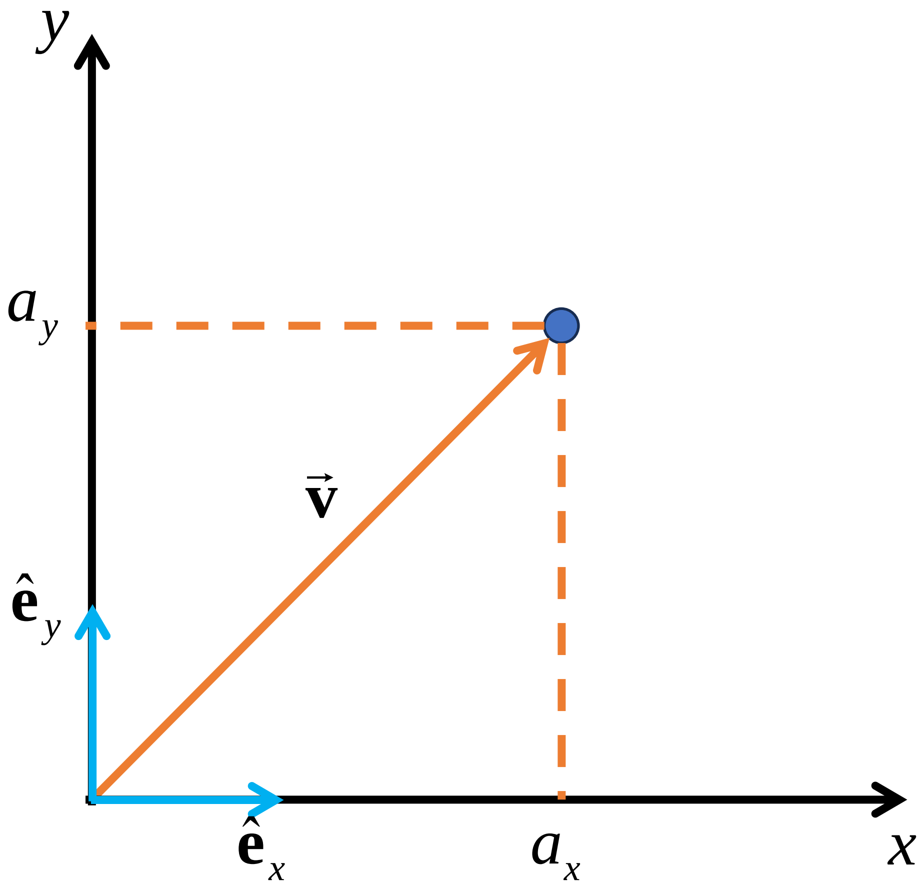
$$\vec{\mathbf{u}} = b_x \hat{\mathbf{e}}_x + b_y \hat{\mathbf{e}}_y$$

Scalar product

$$\vec{\mathbf{v}} \cdot \vec{\mathbf{u}} \equiv a_x b_x + a_y b_y$$

It's a matrix multiplication

$$\vec{\mathbf{v}}^\dagger \cdot \vec{\mathbf{u}} = \begin{pmatrix} a_x & a_y \end{pmatrix} \begin{pmatrix} b_x \\ b_y \end{pmatrix}$$



$$\vec{\mathbf{v}} = a_x \hat{\mathbf{e}}_x + a_y \hat{\mathbf{e}}_y$$

Projection

$$v_x = \hat{\mathbf{e}}_x^\dagger \cdot \vec{\mathbf{v}} = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} a_x \\ a_y \end{pmatrix} = a_x$$

$$v_y = \hat{\mathbf{e}}_y^\dagger \cdot \vec{\mathbf{v}} = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} a_x \\ a_y \end{pmatrix} = a_y$$

State 2 —— $E_2, |\varphi_2\rangle$

State 1 —— $E_1, |\varphi_1\rangle$

Quantum state

$$|\psi\rangle = a_1|\varphi_1\rangle + a_2|\varphi_2\rangle$$

$$= \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

It's analogous to

$$\vec{\mathbf{v}} = a_x\hat{\mathbf{e}}_x + a_y\hat{\mathbf{e}}_y$$

$$= \begin{pmatrix} a_x \\ a_y \end{pmatrix}$$

However, a_1 and a_2 are complex

$$a_n = \alpha + \beta i \quad i^2 = -1$$

State 2 $E_2, |\varphi_2\rangle$

State 1 $E_1, |\varphi_1\rangle$

Quantum state

$$|\psi\rangle = a_1|\varphi_1\rangle + a_2|\varphi_2\rangle$$

$$|\upsilon\rangle = b_1|\varphi_1\rangle + b_2|\varphi_2\rangle$$

Scalar product

$$\langle\psi||\upsilon\rangle \equiv \langle\psi|\upsilon\rangle = a_1^* b_1 + a_2^* b_2$$

$$a_1^* = (\alpha + \beta i)^* = \alpha - \beta i$$

It's a matrix multiplication

$$\langle\psi|\upsilon\rangle = \begin{pmatrix} a_1^* & a_2^* \end{pmatrix} \begin{pmatrix} b_x \\ b_y \end{pmatrix}$$

State 2 $E_2, |\varphi_2\rangle$

State 1 $E_1, |\varphi_1\rangle$

Quantum state

$$|\psi\rangle = a_1|\varphi_1\rangle + a_2|\varphi_2\rangle$$

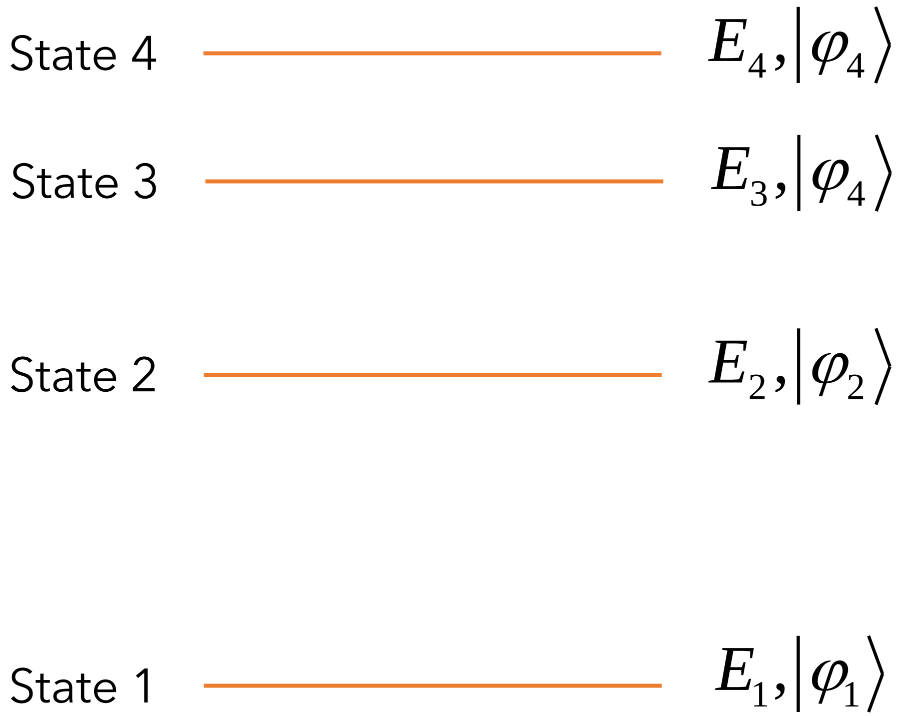
$$= \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

Projection

$$\langle\varphi_1||\psi\rangle \equiv \langle\varphi_1|\psi\rangle$$

$$= \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = a_1$$

$$\langle\varphi_2|\psi\rangle = \begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = a_2$$



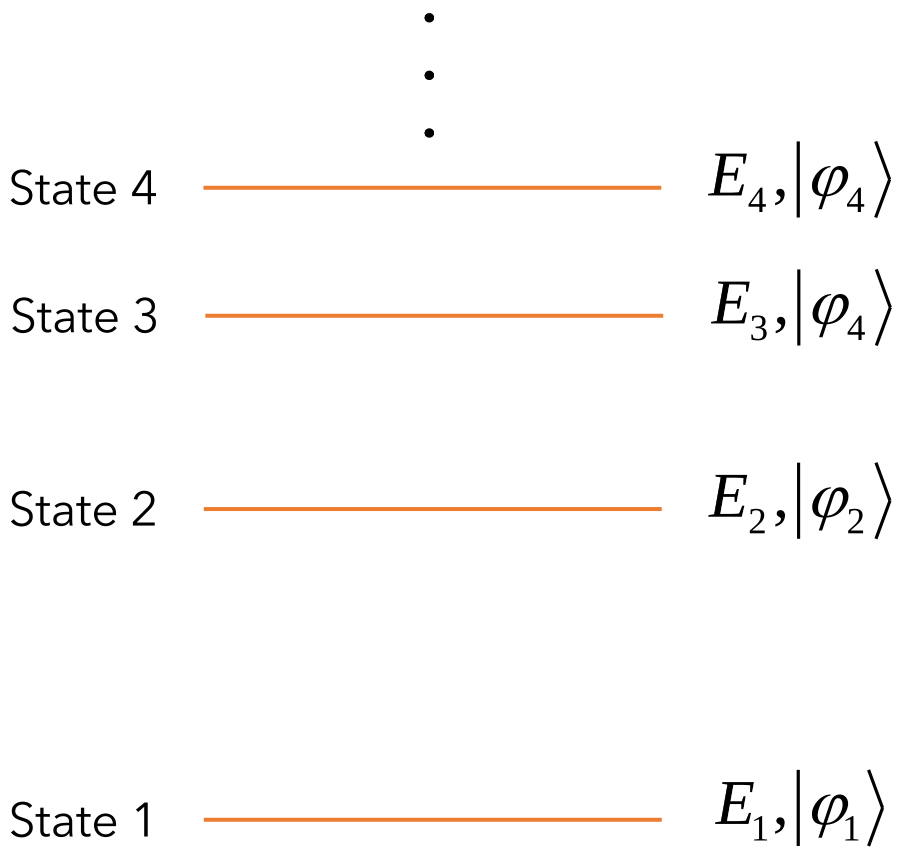
Quantum state

$$|\psi\rangle = a_1|\varphi_1\rangle + a_2|\varphi_2\rangle + a_3|\varphi_3\rangle + a_4|\varphi_4\rangle$$

$$= \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}$$

Projection on state 2, for example

$$\langle\varphi_2|\psi\rangle = \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = a_2$$



Quantum state

$$|\psi\rangle = \sum_n a_n |\varphi_n\rangle$$

Projection on state 2, for example

$$\langle\varphi_2|\psi\rangle = \langle\varphi_2|\sum_n a_n|\varphi_n\rangle$$

$$= \sum_n a_n \langle\varphi_2|\varphi_n\rangle$$

$$= \sum_n a_n \delta_{2n} \qquad \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

$$= 0 + a_2 + 0 + \cdots = a_2$$

Ket	Bra		
$\left|\psi\right\rangle$	$\left\langle\psi\right|$	vector	$\left|\psi\right\rangle = \left(\left\langle\psi\right|\right)^*$

$$c\left|\psi\right\rangle = \left|\varphi\right\rangle$$ vector

$$\left|\psi_1\right\rangle + \left|\psi_2\right\rangle = \left|\varphi\right\rangle$$ vector

$$\left\langle\psi\right| \times \left|\varphi\right\rangle \equiv \left\langle\psi\middle|\varphi\right\rangle = c$$ scalar (simple number)

$$\left|\psi_1\right\rangle \times \left|\psi_2\right\rangle \rightarrow \textit{forbidden}!$$

$$\left|\varphi\right\rangle \times \left\langle\psi\right| \equiv \left|\varphi\right\rangle\left\langle\psi\right| = \hat{O}$$ operator

$$\hat{O}\left|\psi\right\rangle = \left|\varphi\right\rangle$$ vector

Consider

$$|\psi\rangle = \sum_n a_n |\varphi_n\rangle \quad \text{with} \quad \langle\varphi_m|\varphi_n\rangle = \delta_{mn}$$

1. Show that

$$a_m = \langle\varphi_m|\psi\rangle$$

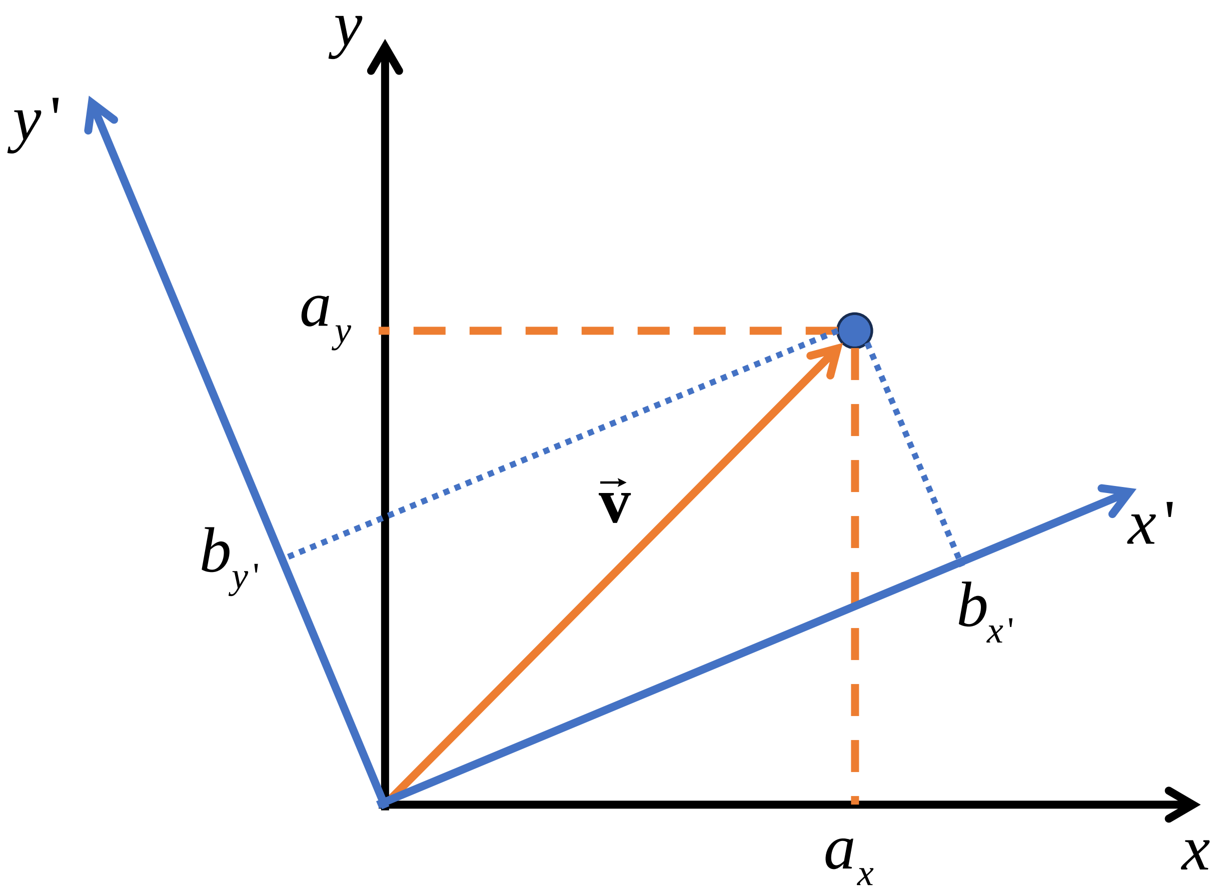
2. Verify that

$$|\psi\rangle = \sum_n |\varphi_n\rangle\langle\varphi_n|\psi\rangle$$

3. Show that

$$\sum_n |\varphi_n\rangle\langle\varphi_n| = 1$$

Basis change



$$\vec{\mathbf{v}} = a_x \hat{\mathbf{e}}_x + a_y \hat{\mathbf{e}}_y$$

$$\vec{\mathbf{v}} = b_{x'} \hat{\mathbf{e}}_{x'} + b_{y'} \hat{\mathbf{e}}_{y'}$$

Consider

$$|\psi\rangle = \sum_n a_n |\varphi_n\rangle$$

Show that this vector is

$$|\psi\rangle = \sum_k b_k |\chi_k\rangle$$

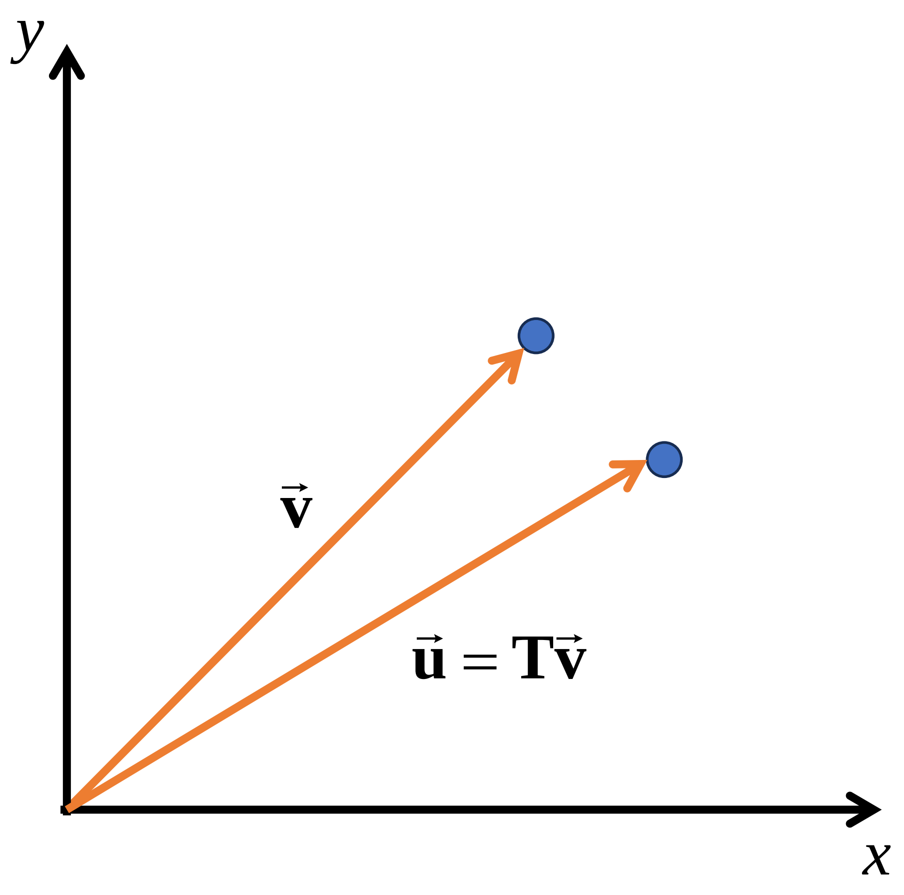
where

$$b_k = \sum_n a_n \langle \chi_k | \varphi_n \rangle$$

Hint: Use

$$\sum_k |\chi_k\rangle\langle\chi_k| = 1$$

Linear transformation



$$\vec{\mathbf{v}} = a_x \hat{\mathbf{e}}_x + a_y \hat{\mathbf{e}}_y$$

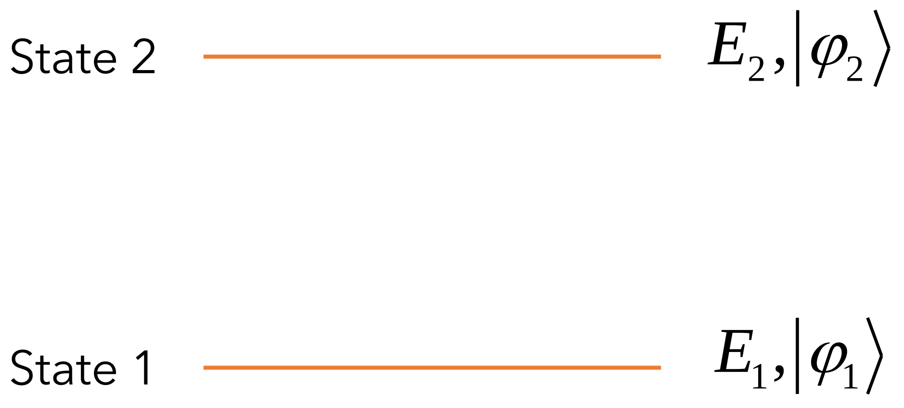
$$\mathbf{T} = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix}$$

Transformation

$$\mathbf{T}\vec{\mathbf{v}} = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \begin{pmatrix} a_x \\ a_y \end{pmatrix}$$

$$= \begin{pmatrix} t_{11}a_x + t_{12}a_y \\ t_{21}a_x + t_{22}a_y \end{pmatrix}$$

$$= \vec{\mathbf{u}}$$



Quantum state

$$|\psi\rangle = a_1|\varphi_1\rangle + a_2|\varphi_2\rangle$$

Operator

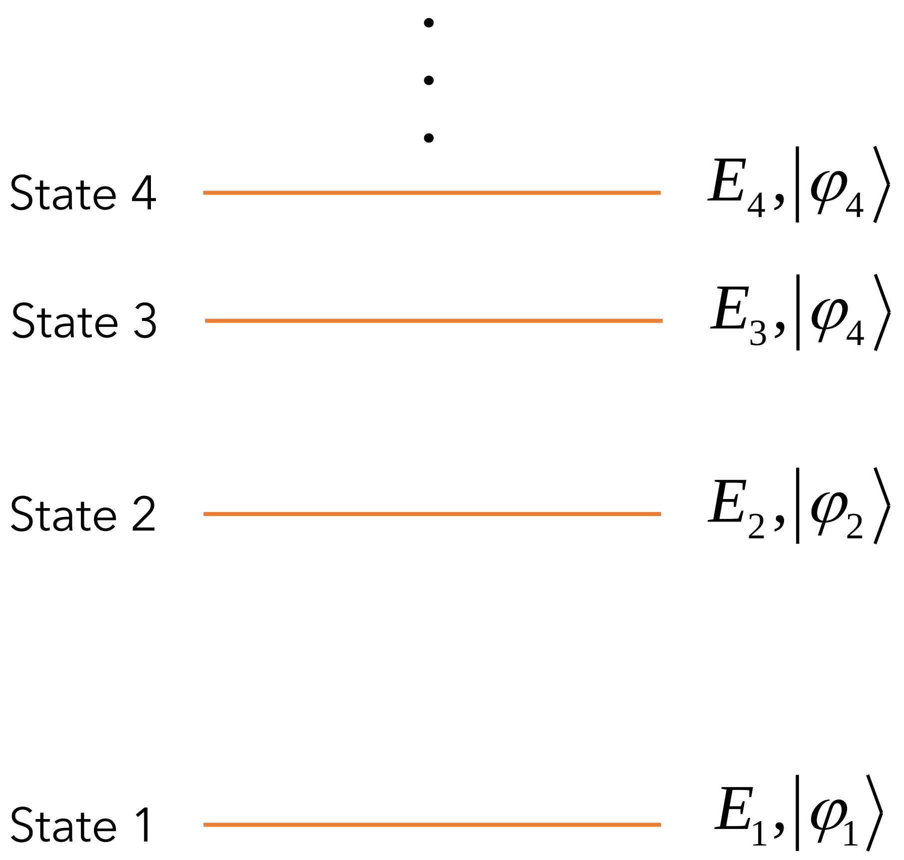
$$\hat{O} = \begin{pmatrix} O_{11} & O_{12} \\ O_{21} & O_{22} \end{pmatrix}$$

Transformation

$$\hat{O}|\psi\rangle = \begin{pmatrix} O_{11} & O_{12} \\ O_{21} & O_{22} \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

$$= \begin{pmatrix} O_{11}a_1 + O_{12}a_2 \\ O_{21}a_1 + O_{22}a_2 \end{pmatrix}$$

$$= |\upsilon\rangle$$



Quantum state

$$|\psi\rangle = \sum_n a_n |\varphi_n\rangle$$

Operation

$$\hat{O}|\psi\rangle = |\upsilon\rangle$$

The transformation is linear if

$$\hat{O}\sum_n a_n |\varphi_n\rangle = \sum_n a_n \hat{O}|\varphi_n\rangle$$

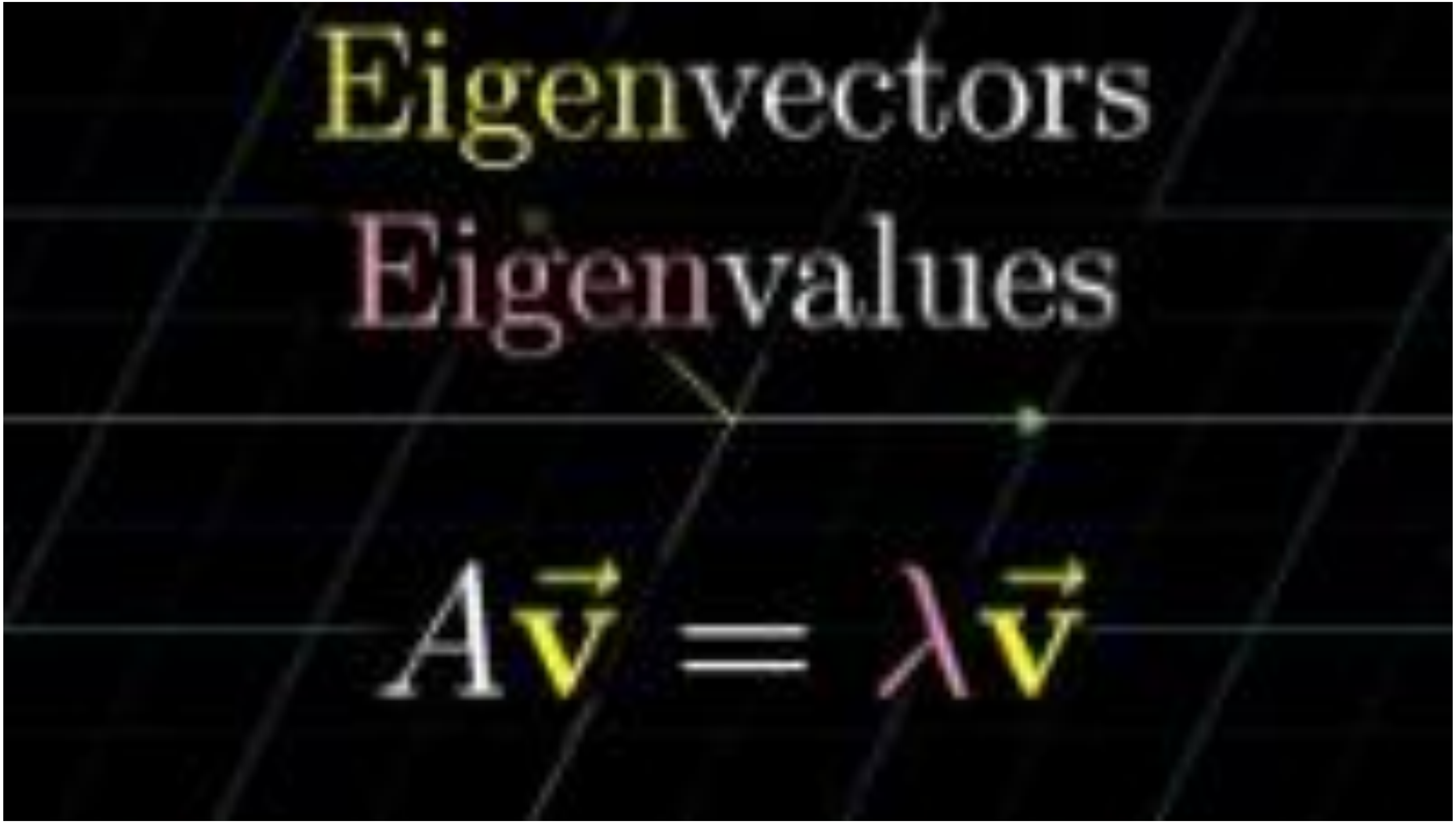
https://youtu.be/kYB8IZa5AuE

Eigenvectors and eigenvalues

Eigenvector

$$\hat{H}|\psi\rangle = E|\psi\rangle$$

Eigenvalue



https://youtu.be/PFDu9oVAE-g?feature=shared

Continuous spectra

Discrete spectrum (Energy, Spin, ...)

$$\hat{A}\left|a_m\right\rangle = a_m\left|a_m\right\rangle$$

$$\left\langle a_m \middle| a_n\right\rangle = \delta_{mn}$$

$$\sum_n \left|a_n\right\rangle\left\langle a_n\right| = 1$$

$$\left|\alpha\right\rangle = \sum_n \left|a_n\right\rangle\left\langle a_n \middle| \alpha\right\rangle$$

Continuous spectrum (Position, Momentum, ...)

$$\hat{\xi}\left|\xi'\right\rangle = \xi'\left|\xi'\right\rangle$$

$$\left\langle \xi' \middle| \xi''\right\rangle = \delta\left(\xi' - \xi''\right)$$

$$\int d\xi' \left|\xi'\right\rangle\left\langle \xi'\right| = 1$$

$$\left|\alpha\right\rangle = \int d\xi' \left|\xi'\right\rangle\left\langle \xi' \middle| \alpha\right\rangle$$

$$|\alpha\rangle = \int d\xi' |\xi'\rangle\langle\xi'|\alpha\rangle$$

$$\alpha(\chi') \equiv \langle\xi'|\alpha\rangle$$ Wave function

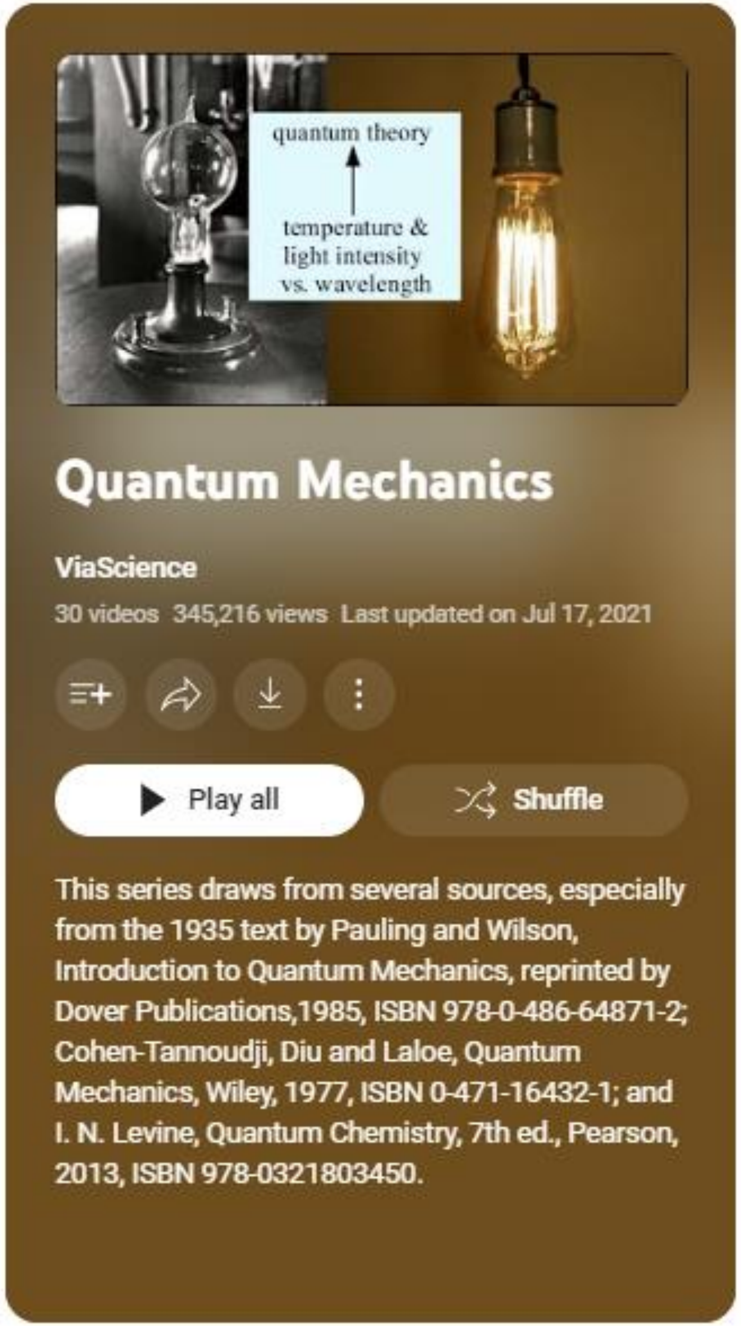
$$|\alpha\rangle = \int d\xi' \alpha(\xi') |\xi'\rangle$$

Example

Position basis $\langle x|\psi\rangle = \psi(x)$ $\quad |\psi\rangle = \int dx\, \psi(x)|x\rangle$

Momentum basis $\langle p|\psi\rangle = \psi(p)$ $\quad |\psi\rangle = \int dp\, \psi(p)|p\rangle$

Recommended:
Quantum mechanics course by ViaScience



https://tinyurl.com/viasciQM

To know more:

Bras and kets

- Sakurai, Modern quantum mechanics, **1994**, Ch 1.

Available for download at:
amubox.univ-amu.fr/s/xXAiMZrDPb9RMRX
Ask me for the password.

For TP2 next week (9 Oct):

- Bring a list of 5 references you think are important for your topic review

- Vijay and I be there to discuss your topic. Come with questions!